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Elastic analysis of unbounded solids using volume integral equation method

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Abstract

A volume integral equation method (VIEM) is used to calculate the plane elastostatic field in an unbounded isotropic elastic medium containing isotropic or anisotropic inclusions subject to remote loading. It should be noted that this newly developed numerical method does not require the Green's function for anisotropic inclusions to solve this class of problems, since only the Green's function for the unbounded isotropic matrix is involved in their formulation for the analysis. A detailed analysis of displacement and stress fields is carried out for isotropic or anisotropic inclusions. The method is shown to be very accurate and effective for investigating the local stresses in composites containing isotropic or anisotropic fibers.

Keywords: Elastic analysis; Volume integral equation method; VIEM; Boundary integral equation method; Finite element method

1. Introduction

Calculation of the stress and strain fields in solids containing multiple inclusions and subjected to external loads is of considerable interest in a variety of engineering applications. A notable example is the stress analysis of damaged fiber reinforced composites that consist of a large number of densely packed fibers with voids or cracks in the matrix. The matrix and the fibers in composites are usually made of isotropic material; however, some of the constituents can be anisotropic. As an example, in SiC/Ti metal matrix composites, the matrix is nearly isotropic, but the SiC fibers have strong anisotropy. A precise knowledge of the deformation and stress fields near interacting isotropic or anisotropic fibers under remote loading can be extremely helpful in predicting the failure and damage mechanisms in the composites.

To our knowledge the only available methods to solve problems of this type are the finite element (FEM) or the boundary integral equation (BIEM) method. However, the finite element method is most effective when the domain of the problem is finite and it is often not possible to separate the influence of the boundary from that of the "microscopic" features of the material on the elastic field. Conventional finite element methods cannot be directly applied to infinite domains. The boundary integral equation method is, in principle, applicable to this class of problems since it can be applied to infinite domains. However, since the Green's function for anisotropic inclusions is involved in the boundary integral equation method and the Green's function for an anisotropic material is much more complex than that for isotropic material, their numerical treatment of the boundary integral equations becomes extremely cumbersome (see, e.g., [1-4]).

In this paper the solution of the general inhomogeneous elastostatic problem is formulated by means of a volume integral equation method (VIEM) for the effective accurate calculation of the stresses and displacements in unbounded isotropic solids in

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placements in unbounded isotropic solids in presence of isotropic or anisotropic inclusions. The recently developed volume integral equation method can be used to calculate the elastic field in composites consisting of an isotropic matrix containing an arbitrary distribution of isotropic or anisotropic fibers or inclusions [5, 6]. In contrast to the boundary integral equation method, the VIEM requires the Green's function for the isotropic matrix only. Moreover, in contrast to FEM, where the full domain needs to be discretized, the VIEM requires discretization of the inclusions only [6].

It should be noted that this newly developed numerical method can also be applied to general two-dimensional elastodynamic as well as elastostatic problems (see, e.g., [7]) for arbitrary geometry and number of inhomogeneities. In the formulation of the method, the continuity condition at each interface is automatically satisfied. Finally, the method takes full advantage of the pre- and post-processing capabilities developed in FEM and BIEM.

In this paper, the VIEM is used to calculate the plane elastostatic field in an unbounded isotropic elastic medium containing isotropic or anisotropic inclusions subject to remote loading. A detailed analysis of stress field at the interface between the matrix and the inclusion is carried out for isotropic or anisotropic inclusions. The accuracy and effectiveness of the new method are examined through comparison with results obtained from analytical and boundary integral equation methods. It is demonstrated that the method is very accurate and effective for investigating the local stresses in composites containing isotropic or anisotropic fibers [8-10].

2. The volume integral equation method (VIEM)

The geometry of the general elastostatic problem is shown in Fig. 1. An unbounded isotropic elastic solid containing inclusions of arbitrary shape is subjected to prescribed loading at infinity. Let c_{ijkl} denote the elastic tensor of the solid. Let $c_{ijkl}^{(1)}$ denote the elastic stiffness tensor of the inclusion and $c_{ijkl}^{(2)}$ those of the unbounded matrix material. The matrix is assumed to be homogeneous and isotropic so that $c_{ijkl}^{(2)}$ is a constant isotropic tensor, while $c_{ijkl}^{(1)}$ is arbitrary, i.e., the inclusions may, in general, be inhomogeneous and anisotropic. The interfaces between the inclusions and

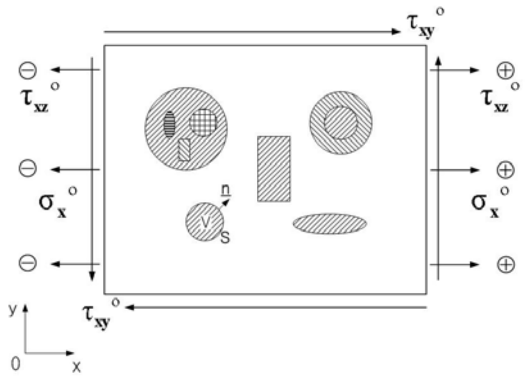


Fig. 1. Geometry of the general elastostatic problem.

the matrix are assumed to be perfectly bonded insuring continuity of the displacement and stress vectors.

It has been shown in (Mal and Knopoff [11]; Lee and Mal [12]) that the elastostatic displacement in the composite satisfies the volume integral equation,

$$u_m(\mathbf{x}) = u_m^0(\mathbf{x}) - \int_R \delta c_{ijkl} g_{i,j}^m(\xi, \mathbf{x}) u_{k,l}(\xi) d\xi, \qquad (1)$$

where the integral is over the whole space, $c_{ijkl} = c_{ijkl}^{(1)} - c_{ijkl}^{(2)}$, and $g_i^m(\xi, \mathbf{x})$ is the static Green's function (or Kelvin's solution) for the unbounded matrix material, i.e., $g_i^m(\xi, \mathbf{x})$ represents the ith component of the displacement at ξ due to unit concentrated force at $\mathbf{x}$ in the mth direction. In equation (1), the summation convention and comma notation have been used and the differentiations are with respect to ξ_i. It should be noted that the integrand is nonzero within the inclusions only, since $c_{ijkl} = 0$, outside the inclusions.

If $\boldsymbol{x} \in \mathbf{R}$ (within the inclusions), then Eq. (1) is an integrodifferential equation for the unknown displacement vector $\mathbf{u}(\boldsymbol{x})$; it can, in principle, be determined through the solution of Eq. (1). An algorithm based on the discretization of Eq. (1) was developed by Lee and Mal [12-13] to calculate numerically the unknown displacement vector $\mathbf{u}(\boldsymbol{x})$ by discretizing the inclusions using standard finite elements. Once $\mathbf{u}(\boldsymbol{x})$ within the inclusions is determined, the displacement field outside the inclusions can be obtained from Eq. (1) by evaluating the integral; and the stress field within and outside the inclusions can also be determined without any difficulty. The details of the numerical treatment of Eq. (1) for plane elastodynamic and elastostatic problems can be found by Lee and Mal [12-13], and will be omitted. In Eq. (1), $g_i^m(\xi, \mathbf{x})$ is the Green's function for the unbounded isotropic matrix material. Thus, the volume integral equation method does not require the use of the Green's function for the anisotropic material of the inclusions. This is in contrast to the boundary integral equation method, where the infinite medium Green's functions for both the matrix and the inclusion are involved in the formulation of the equations. The Green's functions for anisotropic solids can only be obtained in integral forms and their evaluation in the vicinity of the source point is very difficult. Since the numerical implementation of conventional boundary integral equation (BIE) requires the evaluation of the displacements and stresses associated with the Green's function at a large number of points, the method becomes extremely unwieldy if not impossible to apply in even the simplest of model geometries. The present method is free from this problem.

3. Single inclusion problems

3.1 Single isotropic inclusion in the unbounded isotropic matrix

In order to check the accuracy of the volume integral equation method, we first consider plane strain problems for a single isotropic cylindrical inclusion in the unbounded isotropic matrix under uniform remote tensile loading, $\sigma_x^o = \sigma_o$, as shown in Fig. 2. The matrix is assumed to be titanium and the inclusion is SiC fiber of radius 70 μm. The nominal material properties of the constituents are given in Table 1 [14]. The remote applied loads are assumed to be $\sigma_x^o = 143.1$ GPa.

Table 1. Material properties of the constituents of SiC-6/Ti-15-3 composite.

Material	λ (GPa)	μ (GPa)
SiC	176.06	176.06
Ti	67.34	37.88

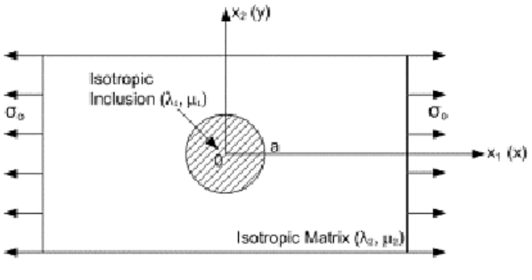


Fig. 2. An isotropic inclusion in unbounded isotropic matrix under uniform remote tensile loading.

Let λ_1, μ_1 denote Lamé constants of the isotropic inclusion, and λ_2, μ_2 Lamé constants of the isotropic matrix, respectively.

For plane strain problems, the volume integral equation (1) becomes

$$\begin{aligned} u_1(x) &= u_1^0(x) \\ &\quad - \int_R \{[\delta(\lambda+2\mu) g_{1,1}^1 u_{1,1} + \delta\lambda g_{1,1}^1 u_{2,2} \\ &\quad + \delta\mu g_{1,2}^1 (u_{1,2} + u_{2,1})] + [\delta(\lambda+2\mu) g_{2,2}^1 u_{2,2} \\ &\quad + \delta\lambda g_{2,2}^1 u_{1,1} + \delta\mu g_{2,1}^1 (u_{1,2} + u_{2,1})]\} d\xi_1 d\xi_2 \end{aligned} \tag{2}$$

and

$$\begin{aligned} u_2(x) &= u_2^0(x) \\ &\quad - \int_R \{[\delta(\lambda+2\mu) g_{1,1}^2 u_{1,1} + \delta\lambda g_{1,1}^2 u_{2,2} \\ &\quad + \delta\mu g_{1,2}^2 (u_{1,2} + u_{2,1})] + [\delta(\lambda+2\mu) g_{2,2}^2 u_{2,2} \\ &\quad + \delta\lambda g_{2,2}^2 u_{1,1} + \delta\mu g_{2,1}^2 (u_{1,2} + u_{2,1})]\} d\xi_1 d\xi_2 \end{aligned} \tag{3}$$

where $u_1(\boldsymbol{x})$, $u_2(\boldsymbol{x})$ are the in-plane displacement components, $\delta(\lambda + 2\mu) = (\lambda_1 + 2\mu_1) - (\lambda_2 + 2\mu_2)$, $\delta\lambda = \lambda_1 - \lambda_2$, and $\delta\mu = \mu_1 - \mu_2$.

In Eqs. (2) and (3), $g_i^m(\xi, \mathbf{x})$ is the Green's function for the unbounded isotropic matrix material and is given by [8, 15]

$$g_i^m = \frac{\lambda+\mu}{4\pi\mu(\lambda+2\mu)} \left[-\frac{\lambda+3\mu}{\lambda+\mu} \ln r \delta_{im} + r_{,i} r_{,m} \right], \tag{4}$$

where $r = |\mathbf{x} - \xi|$ and $i, m = 1, 2$ and λ, μ are the Lamé constants for the unbounded isotropic matrix material. Thus, the VIEM does not require the use of Green's function for the inclusions. This is in contrast to the BIEM, where Green's functions for both the matrix and the inclusions are involved in the formulation of the problem.

Finite element discretization of the inclusions in (2) and (3) results in a two coupled system of linear algebraic equations for the unknown nodal displacements inside the inclusion. Once the displacement field, $\mathbf{u}(\boldsymbol{x})$, within the inclusion is determined, that outside the inclusions can be obtained from equations (2) and (3) by evaluating the integrals. The stress field within and outside the inclusions can also be determined without any difficulty.

Fig. 3 shows a typical discretized model used in the VIEM [16]. A total of 256 standard eight-node quadratic, quadrilateral and six-node quadratic, triangular

Table 2. Normalized stress components within a cylindrical inclusion due to remote loading.

	σ_x / σ_x^o
Exact	1.3167
VIEM	1.3167

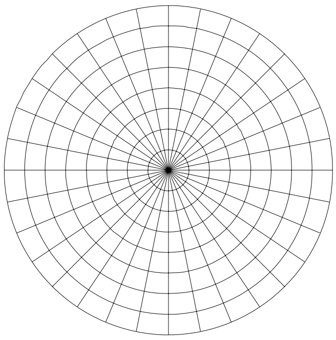


Fig. 3. A typical discretized model in the volume integral equation method.

elements were used in the VIEM. The number of elements, 256, was determined based on a convergence test.

Table 2 shows the comparison between the well-known analytical solution (see, e.g., [17]) and the numerical solution using VIEM. It should be noted that the stress components inside the inclusion are constant. There is excellent agreement between the two sets of results.

The details of the numerical treatment can be found in Lee and Mal [12-13].

3.2 Single orthotropic inclusion in the unbounded isotropic matrix

Consider a single orthotropic cylindrical inclusion in the unbounded isotropic matrix under uniform remote tensile loading, $\sigma_x^o = \sigma_o$, as shown in Fig. 4. Let the coordinate axes $x_1(\boldsymbol{x})$, $x_2(\boldsymbol{x})$, $x_3(\boldsymbol{x})$ be taken parallel to the symmetry axes of the orthotropic material. The matrix is assumed to be titanium and the inclusion is an orthotropic fiber of radius 70 μm. The elastic constants for the isotropic matrix and the orthotropic inclusion are listed in Table 3 [18]. Two different elastic constants for the orthotropic inclusion are considered: in model #1, c_{11} in the inclusion is greater than that in the matrix, and in model #2, c_{11} in the inclusion is smaller than that in the matrix. The remote applied loads are assumed to be $\sigma_x^o = 143.1$ GPa.

Table 3. Material properties of the isotropic matrix and the orthotropic inclusion.

(Unit : GPa)	Isotropic matrix	Orthotropic inclusion	
		#1	#2
λ	67.34	-	-
μ	37.88	-	-
c_{11}	143.10	279.08	13.93
c_{12}	67.34	7.8	0.39
c_{22}	143.10	30.56	1.53
c_{66}	37.88	11.8	0.59

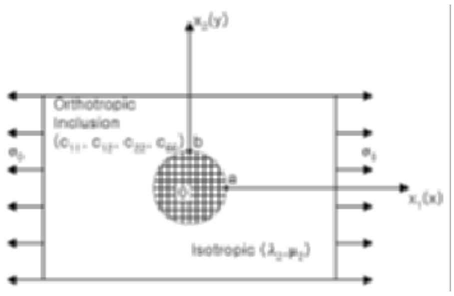


Fig. 4. An orthotropic inclusion in unbounded isotropic matrix under uniform remote tensile loading.

Let c_{11}, c_{12}, c_{22}, c_{66} denote the elastic constants of the orthotropic inclusion, and λ_2, μ_2 the Lamé constants of the isotropic matrix, respectively. The interfaces between the inclusion and the matrix are assumed to be perfectly bonded insuring the continuity of the displacement and stress vectors.

3.2.1 The volume integral equation method (VIEM)

For plane strain problems, the volume integral equation (1) reduces to

$$\begin{aligned} u_1(\mathbf{x}) = u_1^0(\mathbf{x}) & - \int_R \{[\delta c_{11} g_{1,1}^1 u_{1,1} + \delta c_{12} g_{1,1}^1 u_{2,2} \\ & + \delta c_{66} g_{1,2}^1 (u_{1,2} + u_{2,1})] \\ & + [\delta c_{22} g_{2,2}^1 u_{2,2} + \delta c_{12} g_{2,2}^1 u_{1,1} \\ & + \delta c_{66} g_{2,1}^1 (u_{1,2} + u_{2,1})]\} d\xi_1 d\xi_2 \end{aligned} \quad (5)$$

and

$$\begin{aligned} u_2(\mathbf{x}) = u_2^0(\mathbf{x}) & - \int_R \{[\delta c_{11} g_{1,1}^2 u_{1,1} + \delta c_{12} g_{1,1}^2 u_{2,2} \\ & + \delta c_{66} g_{1,2}^2 (u_{1,2} + u_{2,1})] \\ & + [\delta c_{22} g_{2,2}^2 u_{2,2} + \delta c_{12} g_{2,2}^2 u_{1,1} \\ & + \delta c_{66} g_{2,1}^2 (u_{1,2} + u_{2,1})]\} d\xi_1 d\xi \end{aligned} \quad (6)$$

Table 4. Normalized tensile stress component (σ_x / σ_x^o) within the inclusion due to uniform remote tensile loading (σ_x^o).

	Isotropic matrix with orthotropic Inclusion (#1)	Isotropic matrix with orthotropic inclusion (#2)
Exact	1.2388	0.2980
VIEM	1.2389	0.2979
BIEM	1.2397	0.2986

where $u_1(\boldsymbol{x})$, $u_2(\boldsymbol{x})$ are the in-plane displacement components, $\delta c_{11} = c_{11} - (\lambda_2 + 2\mu_2)$, $\delta c_{12} = c_{12} - \lambda_2$, $\delta c_{22} = c_{22} - (\lambda_2 + 2\mu_2)$, and $\delta c_{66} = c_{66} - \mu_2$.

In Eqs. (5) and (6), g_i^m is the Green's function for the unbounded isotropic matrix material. Thus, the volume integral equation method does not require the use of the Green's function for the anisotropic material of the inclusions. This is in contrast to the boundary integral equation method, where the infinite medium Green's functions for both the matrix and the inclusion are involved in the formulation of the equations.

Finite element discretization of the inclusions in (5) and (6) results in a two coupled system of linear algebraic equations for the unknown nodal displacements inside the inclusion. Once the displacement field, $\mathbf{u}(\mathbf{x})$, within the inclusion is determined, that outside the inclusions can be obtained from equations (5) and (6) by evaluating the integrals. The stress field within and outside the inclusions can also be determined without any difficulty.

A total of 256 standard eight-node quadratic, quadrilateral and six-node quadratic, triangular elements in Fig. 3 were used in the VIEM. The number of elements, 256, was determined based on a convergence test.

Table 4 shows the comparison between the analytical solution [9, 10] and the numerical solution by using VIEM. It should be noted that, as expected, the stress components inside the inclusion are constant. There is excellent agreement between the two sets of results [7].

The details of the numerical treatment can be found in Lee and Mal [12, 13].

3.2.2 The boundary integral equation method (BIEM)

The integral equation on the outer surface S_+ of the anisotropic inclusion can be expressed as (Banerjee [8]; Rizzo et al. [15])

$$u_m(\mathbf{x}) = u_m^o(\mathbf{x}) + c_{ijkl}^{(M)} \int_{S_+} [g_{k,l}^{m(M)}(\xi,\mathbf{x})u_i(\xi) - g_i^{m(M)}(\xi,\mathbf{x})u_{k,l}(\xi)]n_j ds \tag{7}$$

while for the interior surface S_-

$$u_m(\mathrm{x}) = -c_{ijkl}^{(I)} \int_{S_-} [g_{k,l}^{m(I)}(\xi,\mathbf{x})u_i(\xi) - g_i^{m(I)}(\xi,\mathbf{x})u_{k,l}(\xi)]n_j ds \tag{8}$$

In equations (7) and (8) n is the outward unit normal to S_+, and the superscripts (M) and (I) indicate that the quantities involved are for the isotropic matrix and the inclusions, respectively. Eqs. (7) and (8), together with the continuity conditions across $\boldsymbol{S}$, give rise to the boundary integral equation for $\mathbf{u}(\boldsymbol{x})$. When the inclusion becomes a void, the integral equations reduce to the standard boundary integral equation

$$u_m(\mathbf{x}) = u_m^o(\mathbf{x}) + c_{ijkl}^{(M)} \int_S g_{k,l}^{m(M)}(\xi,\mathbf{x})u_i(\xi)n_j ds \tag{9}$$

For plane strain problems in Fig. 4, the integral equations on the outer surface of the orthotropic inclusion can be expressed as

$$\begin{aligned} u_1(\mathbf{x}) &= u_1^0(\mathbf{x}) \\ &- \int_{S_+} [g_1^{1(M)}(\xi,\mathbf{x})t_1(\xi) + g_2^{1(M)}(\xi,\mathbf{x})t_2(\xi) \\ &- T_1^{1(M)}(\xi,\mathbf{x})u_1(\xi) - T_2^{1(M)}(\xi,\mathbf{x})u_2(\xi)]dS(\xi) \\ u_2(\mathbf{x}) &= u_2^0(\mathbf{x}) \\ &- \int_{S_+} [g_1^{2(M)}(\xi,\mathbf{x})t_1(\xi) + g_2^{2(M)}(\xi,\mathbf{x})t_2(\xi) \\ &- T_1^{2(M)}(\xi,\mathbf{x})u_1(\xi) - T_2^{2(M)}(\xi,\mathbf{x})u_2(\xi)]dS(\xi) \end{aligned} \tag{10}$$

where $g_\alpha^{\beta(M)}$ is the Green's function for the unbounded isotropic matrix material and is given in Eq. (4).

The associated tractions, $T_\alpha^{\beta(M)}(\alpha,\beta=1,2)$ and t_α are given by

$$\begin{aligned} T_1^{1(M)} &= (\lambda+2\mu)g_{1,1}^{1(M)}n_1 \\ &+ \lambda g_{2,2}^{1(M)}n_1 + \mu(g_{1,2}^{1(M)} + g_{2,1}^{1(M)})n_2, \\ T_1^{2(M)} &= (\lambda+2\mu)g_{1,1}^{2(M)}n_1 \\ &+ \lambda g_{2,2}^{2(M)}n_1 + \mu(g_{1,2}^{2(M)} + g_{2,1}^{2(M)})n_2, \end{aligned}$$

$$T_2^{1(M)} = (\lambda+2\mu) g_{2,2}^{1(M)} n_2 + \lambda g_{1,1}^{1(M)} n_2 + \mu(g_{1,2}^{1(M)} + g_{2,1}^{1(M)}) n_1,$$
$$T_2^{2(M)} = (\lambda+2\mu) g_{2,2}^{2(M)} n_2 + \lambda g_{1,1}^{2(M)} n_2 + \mu(g_{1,2}^{2(M)} + g_{2,1}^{2(M)}) n_1, \quad (11)$$

and

$$t_1 = (\lambda+2\mu) u_{1,1} n_1 + \lambda u_{2,2} n_1 + \mu(u_{1,2} + u_{2,1}) n_2,$$
$$t_2 = (\lambda+2\mu) u_{2,2} n_2 + \lambda u_{1,1} n_2 + \mu(u_{1,2} + u_{2,1}) n_1. \quad (12)$$

For the interior surface, the equations are

$$u_1(\mathbf{x}) = -\int_{S_-} [g_1^{1(I)}(\xi,\mathbf{x}) t_1(\xi) + g_2^{1(I)}(\xi,\mathbf{x}) t_2(\xi) - T_1^{1(I)}(\xi,\mathbf{x}) u_1(\xi) - T_2^{1(I)}(\xi,\mathbf{x}) u_2(\xi)] dS(\xi)$$
$$u_2(\mathbf{x}) = -\int_{S_-} [g_1^{2(I)}(\xi,\mathbf{x}) t_1(\xi) + g_2^{2(I)}(\xi,\mathbf{x}) t_2(\xi) - T_1^{2(I)}(\xi,\mathbf{x}) u_1(\xi) - T_2^{2(I)}(\xi,\mathbf{x}) u_2(\xi)] dS(\xi) \quad (13)$$

It should be noted that $g_\alpha^{\beta(I)}$ and $T_\alpha^{\beta(I)} (\alpha, \beta = 1,2)$ in Eq. (13) are the Green's function and their associated tractions for the orthotropic inclusions. And t_α is given by

$$t_1 = c_{11} u_{1,1} n_1 + c_{12} u_{2,2} n_1 + c_{66}(u_{1,2} + u_{2,1}) n_2$$
$$t_2 = c_{22} u_{2,2} n_2 + c_{12} u_{1,1} n_2 + c_{66}(u_{1,2} + u_{2,1}) n_1 \quad (14)$$

The Green's functions and their associated tractions for the orthotropic material are available in (see, e.g., [18-21]).

Fig. 5 shows a typical discretized model used in the BIEM. A total of 80 standard quadratic elements were used in the BIEM. The number of elements, 80, was determined based on a convergence test.

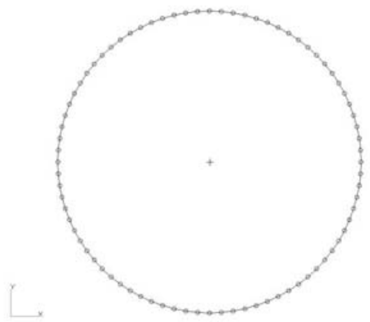


Fig. 5. A typical discretized model in the boundary integral equation method.

Table 4 shows the comparison between the analytical solution [9-10] and the numerical solutions using VIEM and BIEM for the normalized tensile stress component (σ_x / σ_x^o) within the orthotropic inclusion under uniform remote tensile loading (σ_x^o). It can be seen that there is excellent agreement between the three sets of results for all cases considered [7].

However, in general, the Green's function for an anisotropic material is much more complex than that for isotropic materials. Therefore, the numerical implementation of the boundary element method for solving anisotropic inclusion problems becomes extremely cumbersome [4].

3.2.3 Numerical formulation

The volume integral equation involves only $g_\alpha^{\beta(M)}$ and $T_\alpha^{\beta(M)}$ for the isotropic matrix, while the boundary integral equation involves $g_\alpha^{\beta(I)}$ and $T_\alpha^{\beta(I)}$ for the anisotropic inclusions in addition to these. Furthermore, the singularities in VIEM are weaker (integrable) than those in BIEM, where they are of the Cauchy type. We have used the direct integration scheme as introduced by [22-24] after suitable modifications to handle the singularities; a description of the modified method used in the discretization of the volume integral equation is given by Lee and Mal [12, 13].

4. Multiple inclusion problems

For multiple isotropic or anisotropic inclusions, the volume integral equation method is easier and more convenient to apply than the boundary integral equation method. Since the continuity condition at each interface is automatically satisfied in the volume integral equation formulation, it is not necessary to apply continuity conditions at each interface. Also, there is no change in the basic formulation from the single inclusion case. In the host medium, the contrasts in the elastic tensor of the inclusions and the matrix vanish, so that, it is necessary to discretize the isotropic or anisotropic inclusions only. Furthermore, the method is not sensitive to the geometry of the inclusions. The details of the numerical treatment and illustrative examples of problems involving multiple isotropic inclusions can be found by Lee and Mal [12].

4.1 Multiple isotropic inclusions in unbounded isotropic matrix

In order to analyze multiple inclusion interactions, we first consider plane strain problems for multiple isotropic cylindrical inclusions in the unbounded isotropic matrix under uniform remote tensile loading, $\sigma_x^o = \sigma_o$, as shown in Fig. 6. Square packing of (a) 9 inclusions and (b) 25 inclusions is considered with a volume concentration, c = 0.35. The square packing sequence leads to a fiber separation distance d = 2.996a (a: radius of each fiber) for c = 0.35. Fig. 7 shows a typical discretized model used in the VIEM. The standard eight-node quadrilateral and six-node triangular elements were used in the VIEM. In the unbounded isotropic matrix $\delta(\lambda + 2\mu)$, and $\delta\mu$ vanish, so that it is necessary to discretize the isotropic inclusions only. The total number of elements used in VIEM was 2,304. The elastic constants for the materials of the isotropic matrix and the isotropic inclusion are listed in Table 1.

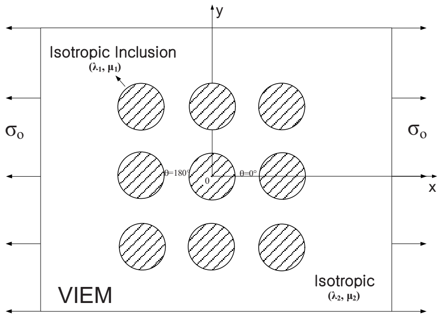


Fig. 6. Multiple isotropic cylindrical inclusions in unbounded isotropic matrix under uniform remote tensile loading.

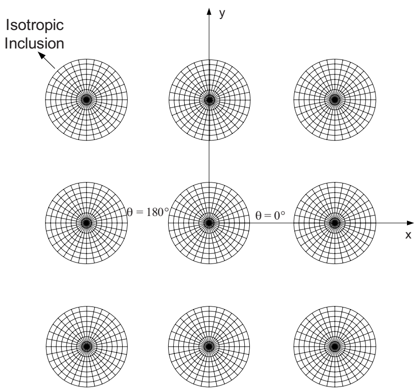


Fig. 7. A typical discretized model in the volume integral equation method for square array of inclusions.

Fig. 8 shows the normalized tensile stress component (σ_x / σ_x^o) at the interface between the matrix and the central inclusion for models containing a single inclusion and two different numbers (9 and 25) of square array of inclusions (θ = 0 ~ 180°). The interaction effect of a square array of inclusions on the normalized tensile stress component at the interface between the matrix and the central inclusion appears to be relatively small at this concentration.

4.2 Multiple orthotropic inclusions in unbounded isotropic matrix

In order to investigate the difference between multiple isotropic-inclusion and anisotropic-inclusion interactions, we next consider plane strain problems for multiple orthotropic cylindrical inclusions in the unbounded isotropic matrix under uniform remote tensile loading, $\sigma_x^o = \sigma_o$, as shown in Fig. 9. Square packing of (a) 9 inclusions and (b) 25 inclusions is considered with a volume concentration, c = 0.35. The square packing sequence leads to a fiber separation distance d=2.996a (a: radius of each fiber) for c= 0.35. Fig. 7 shows a typical discretized model used

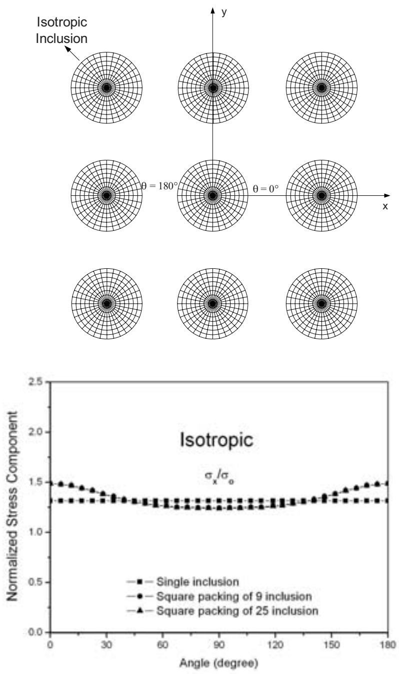


Fig. 8. Normalized tensile stress component (σ_x / σ_x^o) at the interface between the central isotropic inclusion and the isotropic matrix under uniform remote tensile loading. Results are almost the same for larger number of inclusions.

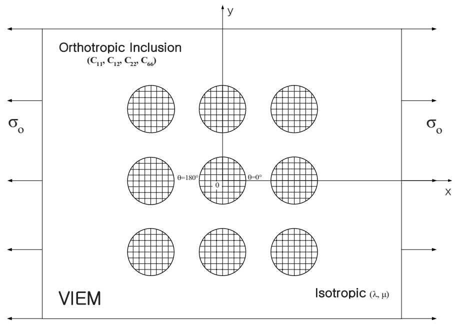


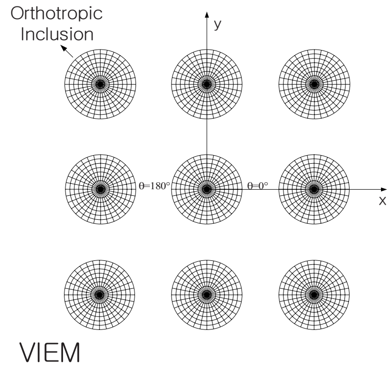
Fig. 9. Multiple orthotropic cylindrical inclusions in unbounded isotropic matrix under uniform remote tensile loading.

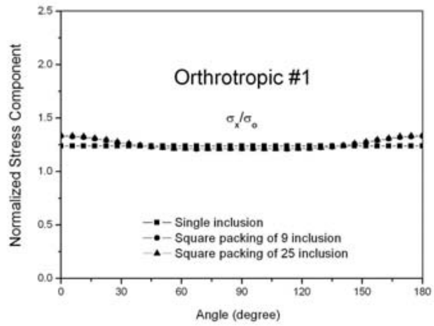
in the VIEM. In the unbounded isotropic matrix, δc_{11}, δc_{12}, δc_{22}, and δc_{66} vanish, so that it is necessary to discretize the orthotropic inclusions only. The total number of elements used in VIEM was 2,304. The elastic constants for the materials of the isotropic matrix and the orthotropic inclusion are listed in Table 3. The remote applied loads are assumed to be σ_x^o = 143.1 GPa.

Fig. 10 shows the normalized tensile stress component (σ_x / σ_x^o) at the interface between the matrix and the central inclusion [model Orthotropic #1(a) and model Orthotropic #2(b)] for models containing a single inclusion and two different numbers (9 and 25) of square array of inclusions (θ = 0 ~ 180°). The interaction effect of a square array of inclusions on the normalized tensile stress component at the interface between the matrix and the central inclusion appears to be small at this concentration [25].

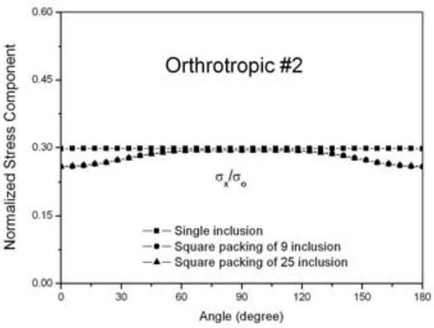
5. Concluding remarks

The volume integral equation method is applied to the calculation of the plane elastostatic field in an unbounded isotropic elastic medium containing isotropic or anisotropic inclusions subject to remote loading. The main advantage of this technique over those based on finite elements is that it requires discretization of the inclusions only in contrast to the need to discretize the entire domain. It is similar to the boundary integral equation method except for the presence of the volume integral over the inclusions instead of the surface integrals over the two sides of the interface. If the medium contains a small number of (isotropic) inclusions, this method may not have any advantage over BIEM. However, in the presence of multiple non-smooth inclusions, the BIEM numerical treatment becomes cumbersome. Since standard finite elements are used in the VIEM, it is very easy and convenient to handle multiple non-smooth inclusions. In elastostatic problems involving multiple anisotropic inclusions, BIEM numerical treatment becomes extremely difficult since the Green's function for an anisotropic material is much more complex than that for isotropic material. How-





(a)



(b)

Fig. 10. Normalized tensile stress component (σ_x / σ_x^o) at the interface between the central orthotropic inclusion and the isotropic matrix under uniform remote tensile loading results are almost the same for larger numbers of inclusions.

ever, the volume integral equation method is free from this problem.

Therefore, through the analysis of plane elastostatic problems in unbounded isotropic matrix with multiple isotropic or orthotropic inclusions, it is established that the VIEM is very accurate and effective for investigating effects of isotropic or general anisotropic fiber packing on stresses in composites containing arbitrary geometry and multiple isotropic or general anisotropic inclusions.

Finally, the formulations developed in this paper can be used to calculate the static stress intensity factors and other quantities of practical interest in realistic models of materials containing strong heterogeneities.

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